

Multi-party zero-error classical channel coding with entanglement

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Abstract

We consider two entanglement-assisted communication problems in which multiple parties have access to one-way classical noisy channels. We focus on the zero-error regime, where the problems can be studied in graph theoretical terms.

The first problem we study is the compound channel, where one sender needs to communicate one message to multiple receivers. We show that the entanglement-assisted capacity of a compound channel converges to the classical one as the number of receivers goes to infinity. We give an upper bound on the number of receivers needed for the convergence, related to the number of channel outputs. On the other hand, we exhibit a class of channels for which entanglement gives an advantage over the classical setting if the number of receivers is fixed.

The second problem we consider features multiple collaborating senders and one receiver. Classically, cooperation between the senders might allow them to communicate on average more bits than the sum of their individual capacities. We show that whenever a channel allows single-sender entanglement-assisted advantage, then the advantage extends also to the multi-sender case. Furthermore, we show that a classical equality regarding a fixed number of uses of a channel with multiple senders is violated in the entanglement-assisted setting.

1 Introduction

We study whether sharing a quantum entangled state improves the zero-error communication in two multi-party scenarios where parties communicate through one-way classical noisy channels.

Let us first describe the classical two-party scenario. Suppose Alice wants to send a message to Bob but they can communicate only through a one-way noisy channel. How much information can she send to him on average, such that Bob learns Alice's message with zero probability of error? Since the introduction of this problem by Shannon [Sha56], several generalizations to multi-party settings have been proposed and a large research area in the intersection of information theory and combinatorics has developed (see [KO98] for a survey).

More recently, Cubitt et al. [CLMW10] introduced the entanglement-assisted version of the problem for two parties: how much data can Alice communicate to Bob with zero error when they are connected through a one-way classical noisy channel and share an entangled quantum state?

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The authors of [LMM⁺12] and [BBG12] exhibit examples of channels for which sharing an entangled state improves the zero-error communication capacity. Two parties can communicate strictly more if they share a certain entangled state than what they could do in the classical case.

Continuing this line of research, we investigate the effect of quantum entanglement in two multi-party zero-error communication scenarios.

In the first situation we study, one sender is connected through identical classical channels to multiple receivers. This is known as a compound channel. Here we prove that entanglement might improve the communication only up to a fixed number of receivers. Indeed, if the number of receivers is greater than a certain threshold (which depends uniquely on the channel), we show that entanglement does not help. However, when there are a constant number of receivers, we can build a channel (which will be an element of a family studied in [BBG12]) for which there is a separation between the entanglement-assisted and classical setting.

In the second problem, multiple senders cooperate to communicate with a single receiver through identical classical channels. Here we show that there exist channels for which entanglement increases the amount the communication that can be sent, independently from the number of senders. Moreover, entanglement enables a violation of a classical equality regarding a fixed number of uses of a channel.

The rest of the paper is organized as follows. In Section 2 we introduce the basic notation and the two-party problem. In Section 3 we study the entanglement-assisted one-sender and multiple-receiver situation. In Section 4 we present the effect of entanglement when there are multiple cooperating senders and one receiver. Section 5 contains the conclusions and some open questions.

2 Preliminaries

2.1 Notation and basics of graph theory

We denote with $[n]$ the set $\{1, \dots, n\}$, with $\Pi(n)$ the symmetric group over $[n]$, with $\delta_{i,j}$ the Kronecker delta function ($\delta_{i,j} = 0$ if $i \neq j$ and $\delta_{i,j} = 1$ if $i = j$) and with I the identity matrix. Consider a positive semidefinite operator ρ that acts on a bipartite finite dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. We denote with $\text{Tr}_A(\rho)$ the partial trace of ρ over the subspace \mathcal{H}_A . Moreover suppose ρ acts on a finite dimensional ℓ -partite Hilbert space $\mathcal{H}^{\otimes \ell}$ which we denote as $B_1 \otimes B_2 \otimes \dots \otimes B_\ell$. With $\text{Tr}_{B_{-k}}(\rho)$ we denote the partial trace of ρ over all the subspaces but the k -th one, *i.e.*, $\text{Tr}_{B_{-k}}(\rho) = \text{Tr}_{B_1, \dots, B_{k-1}, B_{k+1}, \dots, B_\ell}(\rho)$. Throughout the paper, the logarithms are in base two and the graphs are assumed to be undirected and simple.

For any graph G , we denote with $V(G)$ and $E(G)$ its vertex and edge set. If two vertices $u, v \in V(G)$ are equal or adjacent we write $u \simeq v$, analogously we denote $u \sim v$ if they are distinct and adjacent. The complement graph of G , denoted by \overline{G} , is the graph on the same vertex set as G where two distinct vertices are adjacent if and only if they are non-adjacent in G . An independent set of G is a subset of $V(G)$ that contains only pairwise non-adjacent vertices. The *independence number*, $\alpha(G)$, is the maximum cardinality of an independent set of G . A coloring is a partition of the vertex set in independent sets. The *chromatic number*, $\chi(G)$, is the minimum cardinality of a coloring. A clique is a set of pairwise adjacent vertices. The *edge clique cover number*, $\theta_e(G)$, is the smallest number of cliques that together cover all the edges of the graph G . We denote with $\theta'_e(G)$ the edge clique cover number of G plus the number of isolated vertices of G . An *orthogonal representation* of a graph G is a map from the vertex set into the d -dimensional unit sphere such

that adjacent vertices are mapped to orthogonal vectors. The minimum dimension d for which such a representation exists is denoted as $\xi(G)$.

The graph K_t is the complete graph on t vertices, where every pair of distinct vertices is adjacent. The *orthogonality graph* Ω_k has vertex set all the vectors in $\{\pm 1\}^k$ and two vertices are adjacent if and only if the corresponding vectors are orthogonal.

The *strong product graph* of G and H is denoted by $G \boxtimes H$. It has vertex set $V(G) \times V(H)$ (where \times denotes the Cartesian product) and a pair of distinct vertices ux, vy is adjacent if $u \simeq v$ in G and $x \simeq y$ in H . The n -th *strong graph power* of G , denoted $G^{\boxtimes n}$, is the strong product graph of n copies of G . Its vertex set is the Cartesian product of n copies of $V(G)$ and the pair of distinct vertices $(u_1, \dots, u_n), (v_1, \dots, v_n)$ is an edge in $G^{\boxtimes n}$ if and only if $u_i \simeq v_i$ for all $i \in [n]$. We denote with G^{+t} the disjoint union of t copies of the graph G , i.e., $V(G^{+t}) = V(G) \times [t]$ and the pair ui, vj is adjacent if $u \sim v$ in G and $i = j \in [t]$. The *Cartesian product graph* of G with a complete graph K_t , denoted $G \square K_t$, has vertex set $V(G) \times [t]$ and the pair ui, vj is adjacent if $u \sim v$ in G , $i = j \in [t]$ or if $u = v \in V(G)$, $i \neq j \in [t]$.

The Lovász theta number [Lov79] is equal to

$$\vartheta(G) = \max \sum_{u,v \in V(G)} X_{uv} \text{ s.t. } \sum_{u \in V(G)} X_{uu} = 1, X_{uv} = 0 \quad \forall uv \in E(G), X \succeq 0;$$

where $X \succeq 0$ means that X is a positive semidefinite matrix. For any graph G we have $\alpha(G) \leq \vartheta(G) \leq \chi(\overline{G})$. As $\vartheta(G)$ is a positive semidefinite program, it can be computed up to any approximation in polynomial time in the number of vertices $|V(G)|$. Among the useful properties of Lovász theta number, we will use the fact that $\vartheta(G \boxtimes H) = \vartheta(G)\vartheta(H)$ and $\vartheta(\overline{G \boxtimes H}) = \vartheta(\overline{G})\vartheta(\overline{H})$ for every pair of graphs G and H . Moreover ϑ is monotone non-decreasing under taking subgraphs and $\vartheta(K_t) = 1$, $\vartheta(\overline{K}_t) = t$ for all $t \in \mathbb{N}$ (see [KD93] for a survey on the properties of ϑ).

2.2 The zero-error channel capacity

Consider a one-way classical noisy channel \mathcal{N} that connects two parties, Alice and Bob. The channel is fully characterized by its finite input set V , its finite output set W and a probability distribution $\mathcal{N}(\cdot|v)$ for every $v \in V$.

To communicate a message $x \in [m]$, Alice uses an encoding function $C : [m] \rightarrow V$ and sends $C(x)$ through the channel. Bob receives output w with probability $\mathcal{N}(w|C(x))$ and uses a decoding function $D : W \rightarrow [m]$ to get a message $D(w)$. We consider only the zero-error scenario, where $D(w)$ must always be equal to Alice's original message x . What is the maximum size m of a message set that Alice can use? As shown by Shannon [Sha56], this problem can be studied in graph theoretic terms.

To a channel \mathcal{N} with input set V and output set W , we associate a *confusability graph* G , where $V(G) = V$ and the pair $u, v \in V$ is an edge if there exists $w \in W$ such that $\mathcal{N}(w|v)\mathcal{N}(w|u) > 0$. In other words, u and v form an edge if Bob can potentially confuse these inputs. Since we want the communication between Alice and Bob to be zero-error, the largest size m of a message set that they can employ with one use of the channel is the largest set of pairwise non-confusable inputs, i.e., m is the independence number $\alpha(G)$. (Equivalently, Alice can communicate at most $\log \alpha(G)$ bits of information.) The strong graph power $G^{\boxtimes n}$ is the confusability graph of n channel uses. Shannon [Sha56] showed that using the channel multiple times can be on average more efficient

than using it only once. The *Shannon capacity* of a (channel with confusability) graph G ,

$$c(G) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \alpha(G^{\boxtimes n}),$$

is the maximum average number of bits that can be communicated with zero-error. By super-multiplicativity of $\alpha(\cdot)$ and Fekete's Lemma,¹ the Shannon capacity is well-defined and it is also equal to $c(G) = \sup_n \frac{1}{n} \log \alpha(G^{\boxtimes n})$. Computing the Shannon capacity is a notoriously hard problem and its computational complexity is unknown (see for example [AL06]).

The entanglement-assisted variant of this problem, where Alice and Bob share an entangled state, has been introduced in [CLMW10]. Let us describe a protocol for a single use of the channel. To send a message $x \in [m]$ to Bob, Alice performs a measurement $\{A_x^u\}_{u \in V}$ (which might depend on x) on her part of the entangled state, and sends the outcome $u \in V$ through the channel \mathcal{N} . With probability $\mathcal{N}(w|v)$, Bob receives $w \in W$ and might use this information to perform a measurement $\{B_w^y\}_{y \in [m]}$ on his side of the entangled state getting the message y as outcome. Bob outputs y and in the zero-error scenario we require y to be equal to x with zero probability of error. In other words, we want an entangled state $|\psi\rangle_{AB}$ and a measurement with POVM elements $\{A_i^u\}_{i \in [m], u \in V}$ such that

$$\text{Tr}_A(A_i^u \otimes I_B |\psi\rangle\langle\psi|_{AB}) \perp \text{Tr}_A(A_j^v \otimes I_B |\psi\rangle\langle\psi|_{AB}) \text{ for all } i \neq j \in [m], u \simeq v \in V(G),$$

where G is the confusability graph of \mathcal{N} . Setting $\rho_i^u := \text{Tr}_A(A_i^u \otimes I_B |\psi\rangle\langle\psi|_{AB})$, we obtain the following concise definition of the single channel use and we define the entangled Shannon capacity.

Definition 1 ([CLMW10]). For a graph G , the *entanglement-assisted independence number* $\alpha^*(G)$ is the maximum $m \in \mathbb{N}$ such that there exist positive semidefinite operators $\{\rho_i^u : i \in [m], u \in V(G)\}$ and ρ acting on a finite dimensional Hilbert space \mathcal{H} such that

$$\begin{aligned} \text{Tr}(\rho) &= 1, \\ \sum_{u \in V(G)} \rho_i^u &= \rho \quad \forall i \in [m], \\ \rho_i^u \rho_j^v &= 0 \quad \forall i \neq j \in [m], \forall u \simeq v \in V(G). \end{aligned}$$

The *entangled Shannon capacity* is $c^*(G) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \alpha^*(G^{\boxtimes n})$.

Since α^* is super-multiplicative [CLMW10], $c^*(G)$ is well-defined by Fekete's lemma and can be alternatively expressed as $c^*(G) = \sup_n \frac{1}{n} \log \alpha^*(G^{\boxtimes n})$.

The Lovász theta number $\vartheta(G)$ is a known upper bound for the entanglement-assisted independence number, $\alpha^*(G) \leq \lfloor \vartheta(G) \rfloor$ [Bei10, DSW13]. Moreover the multiplicativity of $\vartheta(G)$ under strong graph products [Lov79] implies that $c^*(G) \leq \log \vartheta(G)$.

3 Multiple receivers

In this section we consider a generalization of the channel-coding problem where ℓ receivers want to decode a common message sent by a single sender. This model, known as *compound channel*, was

¹ Super-multiplicativity says that $\alpha(G^{\boxtimes m+m'}) \geq \alpha(G^{\boxtimes m})\alpha(G^{\boxtimes m'})$ for all $m, m' \in \mathbb{N}$. Fekete's Lemma says that if a sequence $(a_m)_{m \in \mathbb{N}}$ is super-additive, $a_{m+m'} \geq a_m + a_{m'} \quad \forall m, m' \in \mathbb{N}$, then the limit of the sequence $(a_m/m)_{m \in \mathbb{N}}$ exists and $\lim_{m \rightarrow \infty} a_m/m = \sup_m a_m/m$. This limit can be infinite, e.g. for $a_m = 2^m$.

introduced independently by [BBT59, Dob59, Wol60] and one of its many applications is message broadcasting. We focus on the zero-error case, where the receivers must decode the message with zero probability of error, and introduce the entanglement-assisted variant. In the classical setting this problem was introduced by [CKS90] as a generalization of the zero-error Shannon capacity (see [Sin09] for a detailed description).

Consider a family of channels $\mathcal{N} = \{\mathcal{N}_1, \dots, \mathcal{N}_\ell\}$ with the same input set V where \mathcal{N}_k connects the sender with the k -th receiver. A common input $v \in V$ is sent to all the receivers and the k -th receiver gets the output w_k according to the distribution $\mathcal{N}_k(w_k|v)$. The goal is for each receiver to retrieve the original input v with zero probability of error. As for the single-channel case, this problem can be treated from a graph-theoretical perspective. To each channel \mathcal{N}_k , we associate a confusability graph $G_k = (V, E_k)$. Note that the family of graphs $\mathcal{G} = \{G_1, \dots, G_\ell\}$ share the same vertex set, since the input set is common. Define $\alpha(\mathcal{G}, n)$ to be the maximum cardinality of a set in V^n which is an independent set in every graph $G_k^{\boxtimes n}$, $G_k \in \mathcal{G}$. Observing that $\alpha(\mathcal{G}, \cdot)$ is super-multiplicative (i.e., $\forall n_1, n_2 \in \mathbb{N}$: $\alpha(\mathcal{G}, n_1 + n_2) \geq \alpha(\mathcal{G}, n_1)\alpha(\mathcal{G}, n_2)$), the Shannon capacity of a family of graphs \mathcal{G} is well-defined as $c(\mathcal{G}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \alpha(\mathcal{G}, n)$. Computing this quantity appears to be hard, since it comprises the computation of the Shannon capacity of a single graph as special case. However, some positive results have been obtained for a slightly different task. Suppose that computing the capacity of every element of the family \mathcal{G} is easy. Is it possible to determine the capacity of the whole family of graphs? An affirmative answer is given in [GKV94].

We study the entanglement-assisted version of this problem focusing on the particular instance where all the channels are equal. Thus, in the rest of the section we assume that all the receivers are connected to the sender through the same channel \mathcal{N} with confusability graph G . This allows us to work directly on G , introducing the notation $\alpha_{1,\ell}(G) = \alpha(\mathcal{G}, 1)$ and $c_{1,\ell}(G) = c(\mathcal{G})$. Note that in the classical case this leads to the trivial situation $\alpha_{1,\ell}(G^{\boxtimes n}) = \alpha(G^{\boxtimes n}) \forall n \in \mathbb{N}$ and thus $c_{1,\ell}(G) = c(G)$ (as any independent set of G is an independent set for the whole family).

The protocol for the entanglement-assisted compound channel is described in Figure 1. As in the single receiver case, the protocol depends only on the confusability graph of the channel and we can define the following quantities.

Definition 2. For a graph G , the *entanglement-assisted compound independence number* with ℓ receivers, $\alpha_{1,\ell}^*(G)$, is defined as the maximum $m \in \mathbb{N}$ such that there exist positive semidefinite operators $\{\rho_i^u, i \in [m], u \in V(G)\}$ and ρ acting on a finite dimensional Hilbert space $\mathcal{H}^{\otimes \ell}$ such that

$$\begin{aligned} \text{Tr}(\rho) &= 1, \\ \sum_{u \in V(G)} \rho_i^u &= \rho \quad \forall i \in [m], \\ \text{Tr}_{B-k}(\rho_i^u) \text{Tr}_{B-k}(\rho_j^v) &= 0 \quad \forall k \in [\ell], \forall i \neq j, \forall u \simeq v \in V(G). \end{aligned}$$

The *entanglement-assisted compound Shannon capacity* with one sender and ℓ receivers is

$$c_{1,\ell}^*(G) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \alpha_{1,\ell}^*(G^{\boxtimes n}).$$

In the following technical lemma we prove that $\alpha_{1,\ell}^*$ is super-multiplicative. This implies by Fekete's lemma that $c_{1,\ell}^*(G)$ is a well-defined quantity and is equivalent to $\sup_n \frac{1}{n} \log \alpha_{1,\ell}^*(G^{\boxtimes n})$.

Lemma 3. For any fixed positive integer ℓ and pair of graphs G and H , $\alpha_{1,\ell}^*(G \boxtimes H) \geq \alpha_{1,\ell}^*(G) \alpha_{1,\ell}^*(H)$.

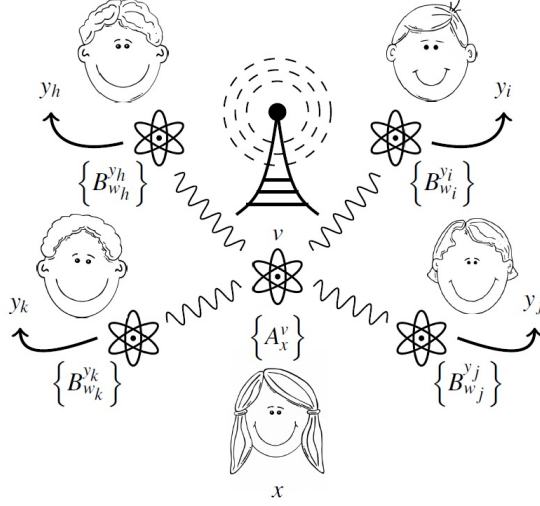


Figure 1: The figure describes an entanglement-assisted compound-channel instance, with a single use of the channels. Alice wants to send the same message x to every Bob, with whom she shares an entangled state. She performs a measurement $\{A_x^v\}$ on her part of the entangled state obtaining v as outcome. This is used as input for all the channels and the k -th Bob receives w_k from \mathcal{N}_k . Each Bob performs a measurement $\{B_{w_k}^{y_k}\}$ (which might depend on w_k) on his part of the entangled state, getting a message y_k as outcome. The protocol works if y_k is equal to x for every k , *i.e.*, every Bob is able to perfectly learn Alice's original message.

Proof. Let $\sigma, \{\sigma_i^u : u \in V(G), i \in [m]\}$ be a collection of positive semidefinite matrices (acting on an ℓ -partite Hilbert space denoted by $B_1 \otimes \cdots \otimes B_\ell$) that are a solution for $\alpha_{1,\ell}^*(G) = m$ and define $(\sigma_k)_i^u := \text{Tr}_{B_{-k}}(\sigma_i^u)$. Moreover let $\tau, \{\tau_j^v : v \in V(H), j \in [n]\}$ be the corresponding collection (acting on an ℓ -partite Hilbert space denoted by $B'_1 \otimes \cdots \otimes B'_\ell$) for $\alpha_{1,\ell}^*(H) = n$ with $n \in \mathbb{N}$ and define $(\tau_k)_j^v := \text{Tr}_{B'_{-k}}(\tau_j^v)$.

We now construct a set $\rho, \{\rho_{i,j}^{u,v} : u \in V(G), v \in V(H), (i,j) \in [m] \times [n]\}$ (acting on an ℓ -partite Hilbert space $\tilde{B}_1 \otimes \cdots \otimes \tilde{B}_\ell$ where $\tilde{B}_k = (B_k \otimes B'_k)$ for every $k \in [\ell]$) which is a feasible solution for $\alpha_{1,\ell}^*(G \boxtimes H)$. For every vertex (u,v) in $G \boxtimes H$ and $(i,j) \in [m] \times [n]$, define

$$\rho_{i,j}^{u,v} := \bigotimes_{k=1}^{\ell} ((\sigma_k)_i^u \otimes (\tau_k)_j^v) \text{ and } \rho = \bigotimes_{k=1}^{\ell} (\text{Tr}_{B_{-k}}(\sigma) \otimes \text{Tr}_{B'_{-k}}(\tau)).$$

Note that $\text{Tr}(\rho) = \text{Tr}(\sigma \otimes \tau) = \text{Tr}(\sigma)\text{Tr}(\tau) = 1$ as we can without loss of generality permute the tensor product spaces when computing the trace. Moreover $\forall i \in [m], j \in [n]$ we have

$$\begin{aligned} \sum_{(u,v) \in V(G) \times V(H)} \rho_{i,j}^{u,v} &= \bigotimes_{k=1}^{\ell} \left(\sum_{u \in V(G)} \sum_{v \in V(H)} (\sigma_k)_i^u \otimes (\tau_k)_j^v \right) \\ &= \bigotimes_{k=1}^{\ell} (\text{Tr}_{B_{-k}}(\sigma) \otimes \text{Tr}_{B'_{-k}}(\tau)) = \rho. \end{aligned}$$

Notice that if (u, v) and (u', v') are distinct and adjacent in $G \boxtimes H$, we have that $u \simeq u' \in V(G)$ and $v \simeq v' \in V(H)$. Then for every $k \in [\ell]$, $(i, j) \neq (i', j')$ and $\{u, v\}, \{u', v'\}$ adjacent in $G \boxtimes H$, we have

$$\begin{aligned} \text{Tr}_{\tilde{B}_{-k}}(\rho_{i,j}^{u,v}) \text{Tr}_{\tilde{B}_{-k}}(\rho_{i',j'}^{u',v'}) &= \left((\sigma_k)_i^u \otimes (\tau_k)_j^v \right) \left((\sigma_k)_{i'}^{u'} \otimes (\tau_k)_{j'}^{v'} \right) \\ &= \left((\sigma_k)_i^u (\sigma_k)_{i'}^{u'} \right) \otimes \left((\tau_k)_j^v (\tau_k)_{j'}^{v'} \right) = 0. \end{aligned}$$

Thus we can conclude that $\alpha_{1,\ell}^*(G \boxtimes H) \geq m \cdot n = \alpha_{1,\ell}^*(G) \alpha_{1,\ell}^*(H)$. \square

3.1 Entanglement does not improve the capacity when the number of receivers goes to infinity

Our first result shows that the entanglement-assisted capacity of a compound channel converges to the classical capacity as the number of receivers goes to infinity. Moreover, this convergence is finite in a number of steps that depends on the number of outputs of the channel. Recall that $\theta'_e(G)$ denotes the edge clique cover number of G plus the number of isolated vertices of G .

Theorem 4. *For any graph G , if $\ell \geq \theta'_e(G)$ then $\alpha_{1,\ell}^*(G) = \alpha(G)$.*

This theorem follows directly from Theorem 8 (Section 3.1.1), where we prove a similar result for non-signaling distributions. Before going to this more general setting, we show how to use Theorem 4 to prove the convergence of the entanglement-assisted Shannon capacity.

Corollary 5. *For any graph G , $\lim_{\ell \rightarrow \infty} c_{1,\ell}^*(G) = c(G)$.*

Proof. First of all, $c(G) = \lim_{\ell \rightarrow \infty} c_{1,\ell}(G) \leq \lim_{\ell \rightarrow \infty} c_{1,\ell}^*(G)$ since the entanglement-assisted strategy is always at least as good as the classical one. Consider the equivalent definitions of $c_{1,\ell}^*$ and c in terms of the supremum. Then,

$$\lim_{\ell \rightarrow \infty} c_{1,\ell}^*(G) = \lim_{\ell \rightarrow \infty} \sup_n \frac{1}{n} \log \alpha_{1,\ell}^*(G^{\boxtimes n}) \leq \sup_n \lim_{\ell \rightarrow \infty} \frac{1}{n} \log \alpha_{1,\ell}^*(G^{\boxtimes n}) = \sup_n \frac{1}{n} \log \alpha(G^{\boxtimes n}) = c(G).$$

\square

3.1.1 Non-signaling capacity

In order to prove Theorem 4, we define a *non-signaling* version of the compound independence number, $\alpha_{1,\ell}^{\text{ns}}(\cdot)$. A non-signaling version of the zero-error channel coding problem was introduced in [CLMW10].

Definition 6. A n -partite probability distribution $P(a_1, a_2, \dots, a_n | x_1, x_2, \dots, x_n)$ is called *non-signaling* if

$$\sum_{a_k} P(a_1 \dots a_k \dots a_n | x_1 \dots x_k \dots x_n) = \sum_{a_k} P(a_1 \dots a_k \dots a_n | x_1 \dots x'_k \dots x_n)$$

for all $k \in [n]$, all inputs a_1, a_2, \dots, a_n and outputs $x_1, \dots, x_k, x'_k, \dots, x_n$.

As every entanglement-assisted strategy is also a non-signaling strategy, the non-signaling capacity of a channel always upper bounds the entanglement-assisted one.

Since we are studying the problem of sending information over a channel with zero-error, we have restricted our attention to the properties of the confusability graph of the channel. However, many channels can have the same confusability graph and, unlike the classical and entanglement-assisted capacities, the non-signaling capacity depends on the particular channel. For our purposes we are interested in the particular channel that minimizes the number of outputs while keeping the same confusability graph. Notice that every output of a channel defines a clique or an isolated vertex in the confusability graph. Therefore, we fix a clique edge covering of the confusability graph of minimum cardinality $\theta_e(G)$ (which might not be unique), we add the isolated vertices to obtain a clique covering of cardinality $\theta'_e(G)$, and we consider the channel that has $\theta'_e(G)$ outputs. In other words, we take a channel which has one output per element of the edge clique covering plus one output per isolated vertex.

The non-signaling protocol for communication through a channel \mathcal{N} with confusability graph G is the following. Alice wants to transmit the message $x \in [m]$. As a resource, Alice and Bob have access to a non-signaling distribution $P(v, y|x, c)$ where Alice inputs x and obtains a vertex $v \in V(G)$ that she sends over the channel. Bob's channel output can be interpreted as a clique (or isolated vertex) $c \subseteq V(G)$ which yields final output $y \in [m]$. The transmission is successful if we always have that $x = y$.

Definition 7. The *non-signaling compound independence number* $\alpha_{1,\ell}^{\text{ns}}(G)$ is the maximum $m \in \mathbb{N}$ such that there exists a non-signaling distribution

$$P(v, j_1, \dots, j_\ell | i, c_1, \dots, c_\ell)$$

between Alice and Bob₁, ..., Bob_ℓ, where $i \in [m]$ and $c_k \subseteq V(G)$ are elements of a fixed clique covering of cardinality $\theta'_e(G)$. Additionally for all $i \in [m]$, $P(v, j_1, \dots, j_\ell | i, c_1, \dots, c_\ell) = 0$ whenever $v \in c_k$ for all $k \in [\ell]$ and there exists a $k' \in [\ell]$ such that $i \neq j_{k'}$.

The last requirement imposes the perfect correctness of the protocol. In fact, every Bob must output the correct message i if he has received a clique c_k that contains Alice's input v .

As already mentioned, for every graph G and $\ell \in \mathbb{N}$, it holds that $\alpha_{1,\ell}^{\text{ns}}(G) \geq \alpha_{1,\ell}^*(G)$ (it also follows from [CLMW10]).

Theorem 8. For all graphs G , if $\ell \geq \theta'_e(G)$ then

$$\alpha_{1,\ell}^{\text{ns}}(G) = \alpha_{1,\ell}^*(G) = \alpha(G).$$

In order to prove the theorem, we use monogamy of non-signaling distributions as derived in [MAG06]. For convenience, we reproduce the definition and result here.

Definition 9. [MAG06] A probability distribution $P(a, b|x, y)$ is called ℓ -shareable with respect to Bob, if there exists an $(\ell + 1)$ -partite probability distribution $Q(a, b_1, \dots, b_\ell | x, y_1, \dots, y_\ell)$ such that:

1. For all permutations $\pi \in \Pi(\ell)$, we have that

$$Q(a, b_{\pi(1)}, \dots, b_{\pi(\ell)} | x, y_{\pi(1)}, \dots, y_{\pi(\ell)}) = Q(a, b_1, \dots, b_\ell | x, y_1, \dots, y_\ell).$$

2. It holds that

$$\sum_{b_2, \dots, b_\ell} Q(a, b_1, \dots, b_\ell | x, y_1, \dots, y_\ell) = P(a, b_1 | x, y_1).$$

Note that if both conditions hold, we have that for all $k \in [\ell]$

$$\sum_{b_1, \dots, b_{k-1}, b_{k+1}, \dots, b_\ell} Q(a, b_1, \dots, b_\ell | x, y_1, \dots, y_\ell) = P(a, b_k | x, y_k). \quad (1)$$

Theorem 10. [MAG06] *Let Y be the set of different values for the input y and suppose $\ell \geq |Y|$. If $P(a, b | x, y)$ is a non-signaling distribution which is ℓ -shareable with respect to Bob, then $P(a, b | x, y)$ admits a local hidden variable model. Formally, there exists a distribution $Q(\lambda)$ over the hidden variables λ as well as local strategies $A(a | x, \lambda)$ for Alice and $B(b | b, \lambda)$ for Bob such that $P(a, b | x, y) = \sum_\lambda Q(\lambda) A(a | x, \lambda) B(b | b, \lambda)$.*

Proof. Assume without loss of generality that $Y = \{1, 2, \dots, |Y|\}$. The idea of the proof is to ask all possible questions $y = 1, 2, \dots, |Y|$ to $|Y|$ different Bob's (which is possible because $\ell \geq |Y|$) and use their answers $b_1, \dots, b_{|Y|}$ to these questions as hidden variables λ .

Assume for now that $\ell = |Y|$. Let us fix the questions to the ℓ Bobs as $y_1 = 1, y_2 = 2, \dots, y_\ell = \ell$ and abbreviate this event with \mathcal{E} . We can then write

$$\begin{aligned} P(a, b | x, y) &= P(a, b | x, y, \mathcal{E}) \\ &= \sum_{\substack{b_1, \dots, b_\ell \\ b_y = b}} Q(a, b_1, \dots, b_\ell | x, y, \mathcal{E}) \\ &\stackrel{(1)}{=} \sum_{b_1, \dots, b_\ell} Q(b_1, \dots, b_\ell | x, y, \mathcal{E}) \cdot Q(a | b_1, \dots, b_\ell, x, y, \mathcal{E}) \cdot \delta_{b, b_y}. \end{aligned}$$

Due to non-signaling, $Q(b_1, \dots, b_\ell | x, y, \mathcal{E}) = Q(b_1, \dots, b_\ell | \mathcal{E}) = Q(\lambda | \mathcal{E})$. Also due to non-signaling, we have $Q(a | b_1, \dots, b_\ell, x, y, \mathcal{E}) = Q(a | b_1, \dots, b_\ell, x, \mathcal{E})$ which defines Alice's strategy $A(a | \lambda, x, \mathcal{E})$. Bob's strategy $B(b | \lambda, y, \mathcal{E})$ is defined by giving the answer $b = b_y$ of the y -th Bob. In summary, we obtain a local-hidden-variable representation of P .

$$P(a, b | x, y) = \sum_\lambda Q(\lambda | \mathcal{E}) \cdot A(a | \lambda, x, \mathcal{E}) \cdot B(b | \lambda, y, \mathcal{E}).$$

In case that $\ell > |Y|$, we observe that ℓ -shareability of $P(a, b | x, y)$ implies $|Y|$ -shareability. Hence, the above proof applies. \square

Proof of Theorem 8. Let $P(v, j_1, \dots, j_\ell | i, c_1, \dots, c_\ell)$ be the optimal non-signaling probability distribution achieving $\alpha_\ell^{\text{ns}}(G)$. We define the following distribution

$$Q(v, j_1, \dots, j_\ell | i, c_1, \dots, c_\ell) := \sum_{\pi \in \Pi(\ell)} \frac{1}{|\Pi(\ell)|} P(v, j_{\pi(1)}, \dots, j_{\pi(\ell)} | i, c_{\pi(1)}, \dots, c_{\pi(\ell)}),$$

which clearly fulfills the first condition of Definition 9. By assumption, we have that for all $i \in [m]$, $P(v, j_1, \dots, j_\ell | i, c_1, \dots, c_\ell) = 0$ whenever $v \in c_k$ for all $k \in [\ell]$ and there is a k' such that $i \neq j_{k'}$. As this condition holds for each pair of Alice and Bob $_k$ individually, it is invariant under permutations

of Bobs. Therefore, the same condition also holds for Q . It follows that Q can also be used to achieve $\alpha_\ell^{\text{ns}}(G)$. We now focus on the marginal distribution $Q(v, j_1|i, c_1)$ between Alice and the first Bob. This distribution is non-signaling and ℓ -shareable by construction where $\ell \geq \theta'_e(G)$, *i.e.*, ℓ is greater or equal to the number of outputs of the specific channel we consider. By Theorem 10, Q admits a local-hidden-variable model. In other words, Alice and the first Bob can achieve the distribution by using classical shared randomness and are therefore unable to transmit more than $\alpha(G)$ messages over the channel, showing that $\alpha_{1,\ell}^{\text{ns}}(G) \leq \alpha(G)$. The claim then follows by combining the inequality with $\alpha(G) = \alpha_{1,\ell}(G) \leq \alpha_{1,\ell}^*(G) \leq \alpha_{1,\ell}^{\text{ns}}(G)$. \square

3.2 Entanglement can improve the capacity for finite number of receivers

In [BBL⁺13], a lower-bound technique for the entanglement-assisted Shannon capacity based on quantum teleportation [BBC⁺93] was presented. In this section, we give a review of this technique while considering the compound-channel scenario. We then use this technique to prove the existence of graphs for which $c_{1,\ell}^*(G) > c_{1,\ell}(G)$ for every fixed $\ell \geq 1$, although the entanglement-assisted capacity approaches the classical one when ℓ increases (as implied by the previous section). Furthermore, we give an upper bound on ℓ for such a separation to exist.

3.2.1 Lower bound by teleportation

We first describe the entanglement-assisted protocol which uses teleportation for the two-party case. Recall that an orthogonal representation of a graph G of dimension d is a map f from $V(G)$ to the complex d -dimensional unit sphere that maps adjacent vertices to orthogonal vectors. Let $t \in \mathbb{N}$ and $\xi(G)$ be the minimum dimension of an orthogonal representation of G . The protocol is as follows:

1. Alice uses the channel t times to send any t -tuple of vertices v_1, \dots, v_t . Bob receives the corresponding outputs w_1, \dots, w_t .
2. Alice teleports the quantum states given by the orthogonal representation $f(v_1), \dots, f(v_t)$ to Bob using the shared (maximally) entangled state. In total, she teleports $t \log \xi(G)$ qubits. Bob now has the rotated versions of the states and needs $2t \log \xi(G)$ classical bits of information to compensate for the Pauli errors.
3. Alice uses the channel one more time to send Bob the $2t \log \xi(G)$ classical bits of correction with zero-error (using an independent set of G).
4. For each $i \in [t]$, Bob uses his output w_i to identify a clique of G that contains v_i . He can then define a projective measurement on the orthogonal representations of the elements of such clique and recover v_i with probability 1 by measuring $f(v_i)$.

The protocol works if t is such that $\alpha(G) \geq \xi(G)^{2t}$. If this is the case, Alice and Bob use a message set of size $m = |V|^t$ by using the channel $t + 1$ times.

The protocol can be easily extended to the multi-receiver case. Suppose that there are ℓ Bobs and that Alice shares an independent maximally entangled state with each of them. With $t + \ell$ uses of the channel, Alice can perfectly communicate one out of $|V(G)|^t$ messages with zero-probability of error to all the Bobs. In fact, with t uses of the channel Alice sends vertices v_1, \dots, v_t . She, then, teleports the orthogonal representations to each Bob individually and uses the channel ℓ

additional times to send the classical bits that complete teleportation. The k -th Bob will consider the $(t + k)$ -th use of the channel as his output and ignore the others. This protocol gives the lower bound:

$$c_{1,\ell}^*(G) \geq \frac{1}{t + \ell} \log |V(G)|^t, \quad (2)$$

where we chose t such that $\alpha(G) \geq \xi(G)^{2t}$ (or equivalently $t \leq \frac{\log(\alpha(G))}{2 \log(\xi(G))}$).

3.2.2 Quarter orthogonality graph

Here, we present a graph family for which entanglement still improves the zero-error capacity with finitely many Bobs. These graphs were introduced in [BBG12] and in [BBL⁺13] they were used to exhibit an infinite family of graphs for which a separation between classical and entanglement-assisted capacity exists.

Definition 11. Let k be an odd positive integer. The *quarter orthogonality graph* Γ_k has the set of $u \in \{\pm 1\}^{k+1}$ such that $u_1 = 1$ and u has an even number of -1 entries as vertex set. The edge set is the pairs of orthogonal vectors.²

For this graph we know that $c(\Gamma_k) \leq 0.846k$ [BBG12] (or more explicitly in [BBL⁺13]). Thus, for all ℓ , $c_{1,\ell}(\Gamma_k) \leq 0.846k$. The map $f : V(\Gamma_k) \rightarrow \mathbb{R}^{k+1}$ such that $f(v) = v/\sqrt{k+1}$ is an orthogonal representation of Γ_k and, hence, $\xi(\Gamma_k) \leq k+1$. So if t is such that $(k+1)^{2t} \leq \alpha(\Gamma_k)$, we have $\xi(\Gamma_k)^{2t} \leq \alpha(\Gamma_k)$ and we can use the teleportation technique from the previous section to communicate $|V(\Gamma_k)|^t$ messages. A simple upper bound on the independence number of Γ_k can be derived considering the set U of vertices that have ones in their last $(k+3)/2$ coordinates. It is easy to see that U is an independent set of cardinality $2^{(k-3)/2}$.

We show that the teleportation lower bound previously described gives a separation between $c_{1,\ell}(\Gamma_k)$ and $c_{1,\ell}^*(\Gamma_k)$ for some $\ell \geq 1$.

Theorem 12. For every odd integer $k \geq 5$ and $\ell \in \mathbb{N}$,

$$c_{1,\ell}^*(\Gamma_k) \geq \frac{t}{t + \ell} (k - 1)$$

with $t = \lfloor \frac{k-3}{4 \log(k+1)} \rfloor$.

Proof. Note that if $t = \lfloor \frac{k-3}{4 \log(k+1)} \rfloor$, from the discussion above we have that $\xi(\Gamma_k)^{2t} \leq (k+1)^{2t} \leq \alpha(\Gamma_k) \leq 2^{(k-3)/2}$. Therefore we can apply Equation (2) and obtain the desired bound, since $|V(\Gamma_k)| = 2^{k-1}$. \square

Corollary 13. If $\ell < \frac{0.144k-1}{0.856k} \lfloor \frac{k-3}{4 \log(k+1)} \rfloor$ then $c_{1,\ell}^*(\Gamma_k) > c_{1,\ell}(\Gamma_k)$.

The result can be obtained by some simple algebraic manipulations. It follows that our lower bound, for $k \approx 1000$ is strictly larger than the classical capacity up to $\ell = 4$, for $k \approx 2000$ up to $\ell = 7$, and ℓ tends to infinity as k goes to infinity.

² Γ_k is called quarter orthogonality graph because it is an induced subgraph of the orthogonality graph Ω_{k+1} (as defined in 2.1) with one quarter of the vertices.

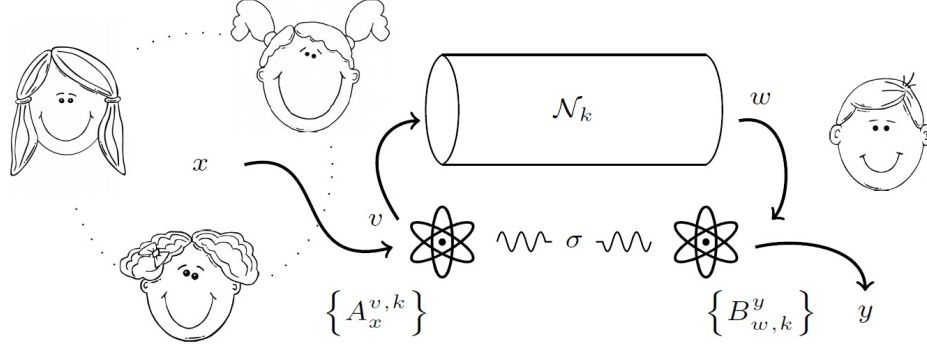


Figure 2: The figure describes an entanglement-assisted multi-sender channel instance, with a single channel use. The Alices can cooperate and the k -th Alice has access to a classical channel \mathcal{N}_k . If the Alices want to communicate the message x to Bob, they perform a measurement $\{A_x^{v,k}\}$ (that might depend on x) in their part of the entangled state. The outcome v, k indicates that the k -th Alice should use her channel \mathcal{N}_k to send input v . Bob receives an outcome w and, by assumption, he knows that channel \mathcal{N}_k has been used for the communication. He can then perform measurement $\{B_w^{y,k}\}$, which might depend on w and k , and outputs y . The protocol works if y is equal to x with zero probability of error.

4 Multiple senders

We now focus on a different zero-error communication scenario: there are multiple senders and a single receiver that might share an entangled state. Suppose there are ℓ Alices, each of whom gets access to a classical channel which connects her to the single Bob. We assume that inputs of one sender can not be confused with inputs from another sender. In other words, the receiver knows which one of the senders send him the message. We are interested in the maximum cardinality of the set of messages that the senders are able to perfectly communicate to the receiver when they are allowed to cooperate.

Let G_k be the confusability graph associated with the channel \mathcal{N}_k . As noticed in [AL07], the confusability graph related to ℓ cooperating senders is given by the disjoint union $G_1 + G_2 + \dots + G_\ell$. Shannon [Sha56] showed that for every pair of graphs G and H if we have $c(G) = \log A$ and $c(H) = \log B$ then $c(G + H) \geq \log(A + B)$ and conjectured that equality holds. However, Alon [Alo98] showed that there exists a pair of graphs for which strict inequality holds. From an information-theoretical perspective, this example implies that if the two senders are allowed to cooperate, they can communicate more efficiently than the sum of their individual possibilities. This result was extended by Alon and Lubetzky [AL07] for a larger number of senders. They showed that it is possible to assign a channel to each sender such that only privileged subsets of senders are allowed to communicate with high capacity.

We extend this problem to the entanglement-assisted setting and focus on the particular case where all the Alices have access to the same channel \mathcal{N} with confusability graph G . The protocol is depicted in Figure 2.

Recall that $G^{+\ell}$ denotes the disjoint union of ℓ copies of the graph G .

Definition 14. For a graph G define the *entanglement-assisted multi-sender independence number* as $\alpha_{\ell,1}^*(G) = \alpha^*(G^{+\ell})$. The *entangled multi-sender Shannon capacity* is $c_{\ell,1}^*(G) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \alpha_{\ell,1}^*(G^{\boxtimes n}) = c^*(G^{+\ell})$.

From the super-multiplicativity of $\alpha^*(G)$, we have that also $\alpha_{\ell,1}^*(G)$ is super-multiplicative and therefore the Shannon capacity $c_{\ell,1}^*(G)$ can be equivalently written as the supremum $\sup_n \frac{1}{n} \log \alpha_{\ell,1}^*(G^{\boxtimes n})$.

In the classical case, we have $\alpha_{\ell,1}(G) = \alpha(G^{+\ell}) = \ell \cdot \alpha(G)$ and $c_{\ell,1}(G) = c(G^{+\ell}) = c(G) + \log \ell$ (the latter equality is also mentioned in [Sha56]).

4.1 Separation between entanglement-assisted and classical multi-sender capacities

We show that for every graph for which there is a separation between the classical and entanglement-assisted capacity, there is also a separation in the multi-sender setting independently from the number of senders (Theorem 16). The same type of result holds when we restrict to single use of the channel (Lemma 15). Note that this is different from the multiple-receiver scenario, where entanglement does not help as the number of receivers goes to infinity.

Lemma 15. *For any graph G such that $\alpha^*(G) > \alpha(G)$, we have $\alpha_{\ell,1}^*(G) > \alpha_{\ell,1}(G)$ for every $\ell \in \mathbb{N}$.*

Proof. Let $\ell \in \mathbb{N}$ and $\{\rho_i^u\}_{i \in [m], u \in V(G)}$, ρ be a solution for $\alpha^*(G) = m$ as in Definition 1. From this solution we will construct a solution for $\alpha_{\ell,1}^*(G)$. Let $\psi = \rho^{\otimes \ell}$ and let the set $\{\psi_j^{u,k}\}_{j \in [m \cdot \ell], u \in V(G), k \in [\ell]}$ be defined as

$$\psi_{i+m \cdot (k-1)}^{u,k} = \frac{1}{\ell} (\rho \otimes \cdots \otimes \rho \otimes \underbrace{\rho_i^u}_{k^{th}} \otimes \rho \otimes \cdots \otimes \rho)$$

for $i \in [m]$. One can check that ψ , $\{\psi_j^{u,k}\}_{j \in [m \cdot \ell], u \in V(G), k \in [\ell]}$ respect all the conditions of Definition 1 and hence, form a solution for $\alpha_{\ell,1}^*(G) = \alpha^*(G^{+\ell})$. Thus, we have

$$\alpha_{\ell,1}^*(G) \geq \ell \cdot m = \ell \cdot \alpha^*(G) > \ell \cdot \alpha(G) = \alpha(G^{+\ell}) = \alpha_{\ell,1}(G).$$

□

Theorem 16. *For any graph G such that $c^*(G) > c(G)$, we have $c_{\ell,1}^*(G) > c_{\ell,1}(G)$ for every $\ell \in \mathbb{N}$.*

Proof. Notice that from the proof of Lemma 15, we have $\alpha_{\ell,1}^*(G) \geq \ell \cdot \alpha^*(G)$ for every $\ell \in \mathbb{N}$ and every graph G . Therefore, we get

$$\begin{aligned} c_{\ell,1}^*(G) &= \sup_{n \in \mathbb{N}} \frac{1}{n} \log \alpha_{\ell,1}^*(G^{\boxtimes n}) \geq \sup_{n \in \mathbb{N}} \frac{1}{n} \log (\ell \cdot \alpha^*(G^{\boxtimes n})) \\ &= \sup_{n \in \mathbb{N}} \frac{1}{n} \log (\alpha^*(G^{\boxtimes n})) + \log(\ell) = c^*(G) + \log(\ell) \\ &> c(G) + \log(\ell) = c_{\ell,1}(G). \end{aligned}$$

□

4.1.1 Violation of a classical equality

As we observed previously, in the classical setting $\alpha_{\ell,1}(G) = \alpha(G^{+\ell}) = \ell \cdot \alpha(G)$ trivially holds for every graph G , $\ell \in \mathbb{N}$ and therefore $c_{\ell,1}(G) = c(G) + \log(\ell)$. This means that in the classical scenario cooperation between the senders does not help in the communication neither with finite number of uses of the channel nor in the asymptotic regime. We now show that in the entanglement-assisted case cooperation between senders can help in the communication for any finite number of uses of the channel. However, we are not able to prove that this improvement gained by cooperation extends also to the asymptotic regime.

In order to prove our result, we need the following definition and properties.

Definition 17. [BBL⁺13] For a graph G , the *entanglement-assisted chromatic number* $\chi^*(G)$ is the minimum $t \in \mathbb{N}$ such that there exist positive semidefinite operators $\{\rho_u^i : i \in [t], u \in V(G)\}$ and ρ acting on a finite dimensional Hilbert space \mathcal{H} such that

$$\begin{aligned} \text{Tr}(\rho) &= 1, \\ \sum_{i \in [t]} \rho_u^i &= \rho \quad \forall u \in V(G), \\ \rho_u^i \rho_v^i &= 0 \quad \forall i \in [t], \forall u \sim v \in V(G). \end{aligned}$$

The entanglement-assisted chromatic number was introduced in [BBL⁺13] to quantify the amount of communication needed in the entanglement-assisted source-coding problem. There, it is shown that $\vartheta(\overline{G}) \leq \chi^*(G)$ and $\chi^*(G^{\boxtimes m}) \leq \chi^*(G)^m$ for every graph G and $m \in \mathbb{N}$.

We will need the following technical lemma. Recall that $G \square K_t$ is the cartesian product graph between G and the complete graph K_t .

Lemma 18. For any graph G , $\chi^*(G) \geq \min\{t \in \mathbb{N} : \alpha^*(G \square K_t) = |V(G)|\}$.

Proof. Let $|V(G)| = n$. For every $t \in \mathbb{N}$, it holds that $\alpha^*(G \square K_t) \leq n$ as $\alpha^*(G \square K_t) \leq \vartheta(G \square K_t) \leq \vartheta(\overline{K}_n \boxtimes K_t) = \vartheta(\overline{K}_n) \cdot \vartheta(K_t) = n$. This chain of inequalities uses that ϑ upper bounds α^* , $\overline{K}_n \boxtimes K_t$ is a subgraph of $G \square K_t$ and ϑ is monotone non-decreasing under taking subgraphs, ϑ is multiplicative under strong graph products and that $\vartheta(K_t) = 1$ and $\vartheta(\overline{K}_t) = t$.

Now suppose $t = \chi^*(G)$ and let $\{\rho_u^i\}_{u \in V, i \in [t]}, \rho$ be the positive semidefinite matrices satisfying Definition 17. We construct a solution for $\alpha^*(G \square K_t)$ as following: for every $(u, i) \in V(G \square K_t)$, $v \in V(G)$ let $\rho_v^{(u,i)} = \delta_{u,v} \rho_u^i$. Then $\sum_{u \in V(G), i \in [t]} \rho_v^{(u,i)} = \sum_{i \in [t]} \rho_v^i = \rho$ and $\rho_v^{(u,i)} \rho_{v'}^{(u',j)} = 0$ if $v \neq v'$ and $\{(u, i), (u', j)\} \in E(G \square K_t)$, i.e., $v \neq v'$ and either $u = u'$, $i \neq j$ or $u \sim u'$, $i = j$. Hence, ρ and $\rho_v^{(u,i)}$ with $(u, i) \in V(G \square K_t)$, $v \in V(G)$ is a feasible solution for Definition 1 and $\alpha^*(G \square K_t) \geq |V(G)| = n$. Together with the first inequality, we derive that if $t = \chi^*(G)$ then $\alpha^*(G \square K_t) = |V(G)|$. From this we can conclude the proof. \square

The following lemma is proven in a more general context in [GL08].

Lemma 19. Let G be a graph such that $\chi^*(G)\alpha^*(G) < |V(G)|$ and let $\chi^*(G) = t$. Then $\alpha^*(G^{+t}) > t \cdot \alpha^*(G)$.

Proof. By Lemma 18, we get $t \cdot \alpha^*(G) < |V(G)| = \alpha^*(G \square K_t) \leq \alpha^*(G^{+t})$ where the last inequality follows from the fact that G^{+t} is a subgraph of $G \square K_t$ and that α^* is monotone non-decreasing under taking subgraphs. \square

Recall from Section 2.1 that the orthogonality graph Ω_k has all the vectors $\{\pm 1\}^k$ as vertex set and two vectors are adjacent if orthogonal. From [RM12], we know that $\vartheta(\Omega_k) = 2^k/k$ and $\vartheta(\overline{\Omega_k}) = k$ if k is a multiple of four. Consider the orthogonal representation $f : V(\Omega_k) \rightarrow \mathbb{R}^k$ with $f(v) = v/\sqrt{k}$ that maps vertices of Ω_k to the unit sphere and adjacent vertices to orthogonal vectors. Since χ^* is upper bounded by the minimum dimension of an orthogonal representation in which all the entries of the vectors have equal moduli [BBL⁺13], we have that $k = \vartheta(\overline{\Omega_k}) \leq \chi^*(\Omega_k) \leq k$, and thus $\chi^*(\Omega_k) = k$, for every k multiple of four.

Now we can show that there exist a graph G and $\ell \in \mathbb{N}$ for which $\alpha_{\ell,1}^*(G) > \ell \cdot \alpha^*(G)$. This implies that, in the entanglement-assisted setting, with one use of a channel ℓ cooperating senders can communicate strictly more than the sum of what they can communicate individually.

Lemma 20. *Let Ω_k be the orthogonality graph with k a multiple of four but not a power of two. Then $\alpha_{k,1}^*(\Omega_k) = \alpha^*(\Omega_k^{+k}) > k \cdot \alpha^*(\Omega_k)$.*

Proof. From the reasoning above we know that $\chi^*(\Omega_k) = k$. Moreover, $\alpha^*(\Omega_k) \leq \lfloor \vartheta(\Omega_k) \rfloor = \lfloor 2^k/k \rfloor < 2^k/k = \vartheta(\Omega_k)$. Using a similar argument as in [RM12], we get that $\chi^*(\Omega_k) \cdot \alpha^*(\Omega_k) < |V(\Omega_k)|$ since

$$\chi^*(\Omega_k) \cdot \alpha^*(\Omega_k) \leq k \cdot \lfloor 2^k/k \rfloor < k \cdot 2^k/k = 2^k = |V(\Omega_k)|.$$

Using Lemma 19 we conclude that $\alpha_{k,1}^*(\Omega_k) = \alpha^*(\Omega_k^{+k}) > k \cdot \alpha^*(\Omega_k)$. \square

With a similar reasoning, we prove that for every finite number of uses of the channel, cooperation between the players improves the entanglement-assisted communication. Let $\alpha_{\ell,1}^*(G, n) := \alpha^*((G^{+\ell})^{\boxtimes n})$ the maximum cardinality of a message set that ℓ Alices can use to communicate perfectly with Bob with n uses of the channel and entanglement. Since the strong graph product distributes over the disjoint union, i.e., $G \boxtimes (H_1 + H_2) = G \boxtimes H_1 + G \boxtimes H_2$ for every G, H_1, H_2 (see for example [HIK11]), we have $\alpha_{\ell,1}^*(G, n) = \alpha^*((G^{+\ell})^{\boxtimes n}) = \alpha^*((G^{\boxtimes n})^{+\ell n})$.

In the next lemma, we show that there exist a graph G and $\ell \in \mathbb{N}$ such that $\alpha_{\ell,1}^*(G, n) > \ell^n \cdot \alpha^*(G^{\boxtimes n})$ for every $n \in \mathbb{N}$. This is equivalent to saying that there exists a channel and a certain number of senders for which cooperation between the senders strictly improves the communication of n channel uses for every $n \in \mathbb{N}$.

Lemma 21. *Let Ω_k be the orthogonality graph with k a multiple of four but not a power of two. Then, $\alpha_{k,1}^*(\Omega_k, n) > k^n \cdot \alpha^*(\Omega_k, n)$ for every $n \in \mathbb{N}$.*

Proof. Using the properties of Lovász theta number presented in Section 2.1, we have that $\vartheta(\Omega_k^{\boxtimes n}) = \vartheta(\Omega_k)^n = \left(\frac{2^k}{k}\right)^n$ and $\vartheta(\overline{\Omega_k^{\boxtimes n}}) = \vartheta(\overline{\Omega_k})^n = k^n$ for every $n \in \mathbb{N}$. Then by sub-multiplicativity of $\chi^*(G)$ [BBL⁺13] and since $\chi^*(\Omega_k) = k$, we have $k^n = \vartheta(\overline{\Omega_k^{\boxtimes n}}) \leq \chi^*(\Omega_k^{\boxtimes n}) \leq \chi^*(\Omega_k)^n = k^n$. This implies that for any integer n ,

$$\alpha^*(\Omega_k^{\boxtimes n}) \leq \lfloor \vartheta(\Omega_k^{\boxtimes n}) \rfloor = \left\lfloor \left(\frac{2^k}{k}\right)^n \right\rfloor < \left(\frac{2^k}{k}\right)^n = \frac{|V(\Omega_k^{\boxtimes n})|}{\chi^*(\Omega_k^{\boxtimes n})}.$$

Applying Lemma 19, we then have that $\alpha^*((\Omega_k^{\boxtimes n})^{+k^n}) > k^n \cdot \alpha^*(\Omega_k^{\boxtimes n})$ for every $n \in \mathbb{N}$. We can conclude that

$$\alpha_{k,1}^*(\Omega_k, n) = \alpha^*((\Omega_k^{\boxtimes n})^{+k^n}) > k^n \cdot \alpha^*(\Omega_k^{\boxtimes n}) = k^n \cdot \alpha^*(\Omega_k, n).$$

\square

5 Conclusions

We have studied the effects of entanglement in two multi-party channel-coding problems. For the compound-channel setting (with one sender and multiple receivers) we have shown that entanglement can only help if the number of receivers is below a certain threshold which depends only on the channel. If there are more receivers, entanglement does not help for zero-error communication. In the second situation, where there are multiple senders and one receiver, we have shown that there are channels for which entanglement always improves the communication.

In both these situations, we assume that the multiple parties have access to identical channels. The first natural question to ask is about the effect of entanglement when the channels are different. Is it possible to prove in the entanglement-assisted setting similar results as the ones obtained by [GKV94] (for multiple receivers) and [AL07] (for multiple cooperating senders)? These problems seem difficult as we only have a limited understanding of the behavior of the parameters α^* and c^* .

A more specific question is related to Theorem 4 in Section 3.1. Can we find a better bound on the number of receivers for which the entanglement-assisted capacity is equal to the classical capacity in the compound channel scenario? The bound we have obtained is the clique edge cover number θ'_e , but this bound is derived using non-signaling distributions and therefore in a more general context than the entanglement-assisted setting.

In general, finding better bounds for the parameter c^* is interesting. Can we find a new general protocol (like the one based on teleportation in Section 3.2.1) that gives a better (or incomparable) lower bound on c^* ? In a similar spirit, can we find an upper bound on c^* which is different from the Lovász theta number? Note that such an upper bound is known for the classical parameter c and it was found by Haemers [Hae78]. A related question is to find a graph G such that $c^*(G) < \log \vartheta(G)$. An approach to this latter problem is to find a pair of graphs, G and H (not necessarily distinct), for which $c^*(G + H) > \log(A + B)$ where $c^*(G) = \log A$ and $c^*(H) = \log B$. Such a result would be in the same spirit as our findings on the parameter α^* described in Section 4.1.1.

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